

Reuse-Growth Sequences: Insertion Bijections and Pattern Avoidance

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Abstract

We introduce *reuse-growth sequences*, a class of statistic-controlled restricted-growth words in which the admissible range increases whenever a previously seen value is reused. The class is singled out by a simple insertion property. If a prefix has length m and contains d distinct values, then it has exactly d extensions by old values and $m + 2 - 2d$ extensions by new values. These are precisely the two insertion counts in the standard generating tree for *simsum* permutations.

Matching the two types of extensions gives a direct recursive bijection between reuse-growth sequences and *simsum* permutations, with

$$\text{des} = d - 1, \quad \text{asc} = \text{rep}.$$

A second recursive bijection maps reuse-growth sequences to 000-avoiding inversion sequences. Consequently,

$$|\mathcal{U}_n| = E_{n+1},$$

with the bijections providing statistic refinements rather than only an equinumeracy.

We then develop the length-three pattern-avoidance theory of the class. All thirteen packed patterns are treated structurally. Six single-pattern classes have closed enumerations. For 102, the ordinary generating function satisfies

$$A = 1 + x(1 - x)A^2 + x^2(1 - x)A^3,$$

from which we obtain a coefficient formula and square-root asymptotics. Moreover, $\mathcal{U}_n(102)$ coincides with the corresponding class of level sequences, so this result also enumerates 102-avoiding level sequences. Four further single-pattern classes admit exact succession rules and are superexponential, while 120, 201, and 210 require ordered used/unused states. For the pair $\{201, 210\}$, these states reduce to three parameters, leading to an explicit algebraic generating function and asymptotics. The resulting avoidance theory differs substantially from that of inversion sequences; in particular, the inversion-sequence Wilf equivalences $101 \sim 110$ and $201 \sim 210$ do not persist for reuse-growth sequences.

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1 Introduction

A permutation is *simsum* if its subword on $\{1, \dots, k\}$ has no three consecutive entries in decreasing order, for every k . Write RS_m for the set of *simsum* permutations of length m . Descents and ascents refer to adjacent comparisons. If $\pi \in \text{RS}_m$ has k descents, the gaps that allow insertion of the new maximum $m + 1$ split into $k + 1$ that preserve the descent count and $m - 2k$ that increase it [10, 11]. The word class studied here has the same two numbers of extensions.

For a finite word w , let $d(w)$ be the number of distinct values in w and put

$$\text{rep}(w) = |w| - d(w).$$

Thus $\text{rep}(w)$ is the number of positions whose value has appeared earlier. A *reuse-growth sequence* is a word of nonnegative integers satisfying

$$w_1 = 0, \quad w_i \leq 1 + \text{rep}(w_1 \cdots w_{i-1}) \quad (i \geq 2).$$

Write $\mathcal{U}_n$ for the set of such words of length n . If a prefix has length m and uses d distinct values, then every used value is still legal. Hence it has

$$d \text{ old-value extensions,} \quad m + 2 - 2d \text{ new-value extensions.}$$

Putting $k = d - 1$ gives exactly the two simsun insertion counts above. Matching old values with descent-preserving gaps and new values with descent-creating gaps, in order, gives a recursive bijection

$$\Phi_n : \mathcal{U}_n \longrightarrow \text{RS}_n, \quad \text{des}(\Phi_n(w)) = d(w) - 1, \quad \text{asc}(\Phi_n(w)) = \text{rep}(w).$$

In particular, $|\mathcal{U}_n| = E_{n+1}$, where E_j is the j th Euler zigzag number.

Prefix bounds defined by word statistics occur in several familiar sequence classes. Mansour and Vajnovszki considered such rules in a general setting [1], and the statistic rep occurs among the Euler–Stirling statistics on ascent sequences [2]. Nearby examples use different information from the prefix. Level sequences use the same prefix rule with rep replaced by the number of adjacent equalities [3, 4]; write $\mathcal{L}_n$ for this class. Ordinary restricted growth functions use the running maximum and encode set partitions [5, 6]. In a reuse-growth sequence, by contrast, every repetition increases the next upper bound, whether or not the repeated value is the current value.

An *inversion sequence* is a word $e = e_1 \cdots e_n$ with $0 \leq e_i < i$; let $\mathcal{I}_n$ be their set. Every reuse-growth sequence is an inversion sequence because the upper bound at position i is at most $i - 1$. The old/new extension split also gives a recursive bijection

$$\Theta_n : \mathcal{U}_n \longrightarrow \mathcal{I}_n(000).$$

Here $\mathcal{I}_n(000)$ consists of inversion sequences in which each value occurs at most twice. An old reuse value is matched with an unused inversion value, and a new reuse value with a value that has occurred exactly once. Let $\text{dist}(e)$ count the distinct *positive* values of e . Then

$$\text{rep}(w) = \text{dist}(\Theta_n(w)), \quad d(w) - 1 = \#\{a : a \text{ occurs twice in } \Theta_n(w)\}.$$

The Euler enumeration of $\mathcal{I}_n(000)$ and a refined bijection from this class to simsun permutations are known [7, 9]. The map Φ_n above is a direct bijection from reuse-growth words to simsun permutations; it does not pass through inversion sequences.

A pattern is *packed* if its distinct values are $0, 1, \dots, k$ for some $k \geq 0$. A word contains a pattern when a subsequence has the same strict comparisons and equalities. Write $\mathcal{U}_n(p)$, $\mathcal{I}_n(p)$, and $\mathcal{L}_n(p)$ for the corresponding avoiding classes. We study the thirteen packed patterns of length three. For inversion sequences, Testart [15] completes the enumeration of the single- and double-pattern cases. The map Θ_n does not preserve pattern avoidance: for example, $\Theta_3(000) = 012$. We prove

$$\mathcal{L}_n(012) = \mathcal{U}_n(012) = \mathcal{I}_n(012), \quad \mathcal{L}_n(102) = \mathcal{U}_n(102).$$

Thus the odd-indexed Fibonacci enumeration for 012 is the known enumeration of the same set of words [7]. The second equality has a different consequence: Mansour explicitly derives the single-pattern level-sequence generating functions for 001 and 012; no generating function for $\mathcal{L}_n(102)$ is given there [4, Section 3]. Hence our 102 result also enumerates the level sequences

avoiding 102. The pattern 021 gives a different comparison. For inversion sequences and reuse-growth sequences, avoidance is equivalent to weak increase of the positive entries, but inversion sequences are counted by the large Schröder numbers [7, 8], while reuse-growth sequences have generating function

$$R_{021}(x) = \frac{1 - 2x^2 - \sqrt{1 - 4x - 4x^2}}{2(1 - x + x^2)}.$$

Here $R_p(x) = \sum_{n \geq 1} |\mathcal{U}_n(p)|x^n$ excludes the empty word. For 102 we use $A_{102}(x) = 1 + R_{102}(x)$. With the empty word included, the inversion-sequence generating function satisfies $A = 1 + (x - x^2)A^3$ [8]; for reuse-growth sequences we obtain

$$A_{102} = 1 + x(1 - x)A_{102}^2 + x^2(1 - x)A_{102}^3.$$

This equation also gives a Lagrange coefficient formula and square-root asymptotics.

The comparison with inversion sequences is not limited to different generating functions. Corteel–Martinez–Savage–Weselcouch proved that the inversion-sequence patterns 101 and 110 are Wilf-equivalent, as are 201 and 210 [7]. The corresponding reuse-growth classes are not Wilf-equivalent: the exact rules below give

$$|\mathcal{U}_6(101)| = 192 \neq 191 = |\mathcal{U}_6(110)|, \quad |\mathcal{U}_6(201)| = 256 \neq 257 = |\mathcal{U}_6(210)|.$$

For 010, 100, 101, and 110, we record the admissible values in increasing order and mark those already used. This gives exact extension rules, and a common family of words gives

$$|\mathcal{U}_n(p)| \geq \lfloor n/2 \rfloor!, \quad p \in \{010, 100, 101, 110\}.$$

For 101, these records can be written as weak compositions, and the number of distinct records at level n is $F_{n+2} - 1$. We also give exact extension tests for 120, 201, and 210. For the pair $\{201, 210\}$, three integers suffice to describe the continuation rule; the resulting functional equation can be solved by two kernel substitutions. This gives an algebraic generating function and an explicit asymptotic formula. The pair $\{120, 210\}$ has a two-parameter succession rule.

Section 2 proves the two insertion bijections and the set-theoretic comparison with inversion sequences. Section 3 treats the six single patterns with closed enumerations. Section 4 gives the exact rules for 010, 100, 101, and 110. Section 5 treats 120, 201, 210 and the two intersections. Section 6 lists the remaining questions.

2 Reuse-growth sequences and two insertion bijections

For a finite word $w = w_1 \cdots w_m$ on the nonnegative integers, define

$$d(w) = |\{w_1, \dots, w_m\}|, \quad \text{rep}(w) = \#\{i \geq 2 : w_i \in \{w_1, \dots, w_{i-1}\}\} = m - d(w).$$

Definition 2.1. A reuse-growth sequence of length n is a word $w = w_1 \cdots w_n$ of nonnegative integers satisfying

$$w_1 = 0, \quad w_i \leq 1 + \text{rep}(w_1 \cdots w_{i-1}) \quad (i \geq 2).$$

The set of these words is denoted by $\mathcal{U}_n$.

For example, $01021 \in \mathcal{U}_5$. The upper bounds at positions 2, 3, 4, 5 are 1, 1, 2, 2. Its entries at positions 3 and 5 reuse values already seen, so $d(01021) = 3$ and $\text{rep}(01021) = 2$. After this prefix, the legal next values are 0, 1, 2, 3: the first three are old and 3 is new.

For $w \in \mathcal{U}_m$, put

$$B(w) = 1 + \text{rep}(w) = m - d(w) + 1.$$

Every value already used lies in $\{0, \dots, B(w)\}$, because the bound is nondecreasing along prefixes. Consequently, when $d = d(w)$, the legal extensions split into

$$d \text{ old values,} \quad m + 2 - 2d \text{ new values.} \quad (1)$$

Here “old” and “new” refer to occurrence anywhere in the prefix, not just to equality with its last entry.

2.1 A direct bijection with simsun permutations

A gap is an insertion position, including the positions before the first entry and after the last. A *double descent* consists of three consecutive entries in decreasing order. A *fertility gap* is a gap in which insertion of the new maximum preserves the simsun property [11].

Theorem 2.2. *For every $n \geq 1$ there is a direct recursive bijection*

$$\Phi_n : \mathcal{U}_n \longrightarrow \text{RS}_n$$

satisfying

$$\text{des}(\Phi_n(w)) = d(w) - 1, \quad \text{asc}(\Phi_n(w)) = \text{rep}(w). \quad (2)$$

Proof. The base case is $0 \leftrightarrow 1$. Suppose recursively that $w \in \mathcal{U}_m$, with $d = d(w)$, corresponds to $\pi \in \text{RS}_m$ with $k = d - 1$ descents. The word has the extensions in (1).

A gap immediately before the top of a descent of π is forbidden: inserting $m + 1$ there produces a double descent beginning with $m + 1$. These are exactly the forbidden gaps. Indeed, the restrictions to $\{1, \dots, j\}$ for $j \leq m$ are unchanged by insertion, and a new double descent in the full permutation can only begin with the new maximum. The k gaps inside the existing descents and the terminal gap preserve the number of descents. Thus there are $k + 1 = d$ descent-preserving fertility gaps. The other fertility gaps create one descent; their number is

$$(m + 1 - k) - (k + 1) = m - 2k = m + 2 - 2d.$$

When $m + 2 - 2d = 0$, both the new-value list and the descent-creating gap list are empty.

Order the old legal word values increasingly and the descent-preserving fertility gaps from left to right, and match them by rank. Independently, match the new legal values in increasing order with the descent-creating fertility gaps from left to right. Appending a word value corresponds to inserting $m + 1$ in its matched gap. An old choice leaves d and the descent count unchanged; a new choice increases each by one. This proves the descent identity inductively.

For the inverse, take $\pi' \in \text{RS}_{m+1}$ and delete its maximum. The remaining permutation π is simsun. The former position of $m + 1$ identifies a fertility gap of π , whose type and left-to-right rank are determined in π . Recover the parent word recursively, then use that type and rank to select its final old or new value. This reverses every step uniquely.

Finally, a permutation of length m has $m - 1$ adjacent comparisons. Hence

$$\text{asc}(\Phi_m(w)) = (m - 1) - (d(w) - 1) = m - d(w) = \text{rep}(w).$$

□

Corollary 2.3. *Let E_j be the Euler zigzag numbers, normalized by*

$$\sum_{j \geq 0} E_j \frac{x^j}{j!} = \sec x + \tan x.$$

Then $|\mathcal{U}_n| = E_{n+1}$. The counts for lengths $1, 2, \dots$ begin

$$1, 2, 5, 16, 61, 272, 1385, 7936, \dots$$

Proof. The number of simsun permutations of length n is E_{n+1} [11]. Apply Theorem 2.2. □

2.2 A bijection with 000-avoiding inversion sequences

An inversion sequence is a word $e = e_1 \cdots e_n$ with $0 \leq e_i < i$. Let $\mathcal{I}_n$ be their set, and let $\mathcal{I}_n(000)$ consist of those in which every value occurs at most twice. This class is Euler-enumerated [7] and has a refined bijection with simsun permutations [9].

Theorem 2.4. *For every $n \geq 1$ there is a recursive bijection*

$$\Theta_n : \mathcal{U}_n \longrightarrow \mathcal{I}_n(000)$$

such that, for $e = \Theta_n(w)$,

$$\text{rep}(w) = \text{dist}(e), \quad d(w) - 1 = \#\{a : a \text{ occurs twice in } e\}, \quad (3)$$

where $\text{dist}(e)$ is the number of distinct positive values of e .

Proof. Set $\Theta_1(0) = 0$. Suppose $w \in \mathcal{U}_m$ corresponds to $e \in \mathcal{I}_m(000)$ and put $d = d(w)$. The induction hypothesis says that e has $m - d + 1$ distinct values when zero is included, and $d - 1$ values occurring twice. Among $0, 1, \dots, m$, therefore, d values are unused and $m + 2 - 2d$ occur once.

Let $O(w)$ and $N(w)$ be the increasing lists of old and new legal reuse values. Let $Z(e)$ be the increasing list of unused values in $\{0, \dots, m\}$, and $S(e)$ the increasing list of values occurring once. Match $O(w)$ with $Z(e)$ by rank and $N(w)$ with $S(e)$ by rank. If a is appended to w , append its matched value to e .

An old word value is matched with an unused inversion value. Thus $d(w)$ stays fixed while $\text{rep}(w)$ increases by one. Zero already occurs in e , so the matched unused value is positive, and $\text{dist}(e)$ also increases by one. A new word value is matched with an inversion value occurring once; it now occurs twice. Both $d(w)$ and the number of double occurrences increase by one, whereas rep and dist do not change. No inversion value is chosen for a third occurrence.

To invert the map, delete the final inversion value y and recursively recover the parent word w from the parent inversion sequence e . If y occurs zero times in e , use its rank in $Z(e)$ to choose the old value of the same rank in $O(w)$. If it occurs once, use its rank in $S(e)$ to choose the corresponding new value in $N(w)$. These are the only two possibilities and give a unique inverse. $\square$

Remark 2.5. Corteel–Martinez–Savage–Weselcouch proved that the number of 000-avoiding inversion sequences with $n - k$ distinct entries equals the number of simsun permutations with k descents [7, Corollary 2]. Since zero occurs in every inversion sequence, this already implies the equidistribution of dist with simsun ascents. Lin and Kim later gave a refined bijection between $\mathcal{I}_n(000)$ and RS_n preserving further statistics [9]. In Theorem 2.2, old and new reuse values are matched directly to the two types of simsun insertion gaps.

2.3 Inclusions and shared avoidance classes

For a word $w = w_1 \cdots w_m$, let $\text{lev}(w) = \#\{i : 2 \leq i \leq m, w_i = w_{i-1}\}$. The level sequences of Callan [3], also called repetition sequences, satisfy $w_1 = 0$ and $w_i \leq 1 + \text{lev}(w_1 \cdots w_{i-1})$ for $i \geq 2$. Their set of length n is $\mathcal{L}_n$.

Proposition 2.6. *For every $n \geq 1$,*

$$\mathcal{L}_n \subseteq \mathcal{U}_n \subseteq \mathcal{I}_n, \quad \mathcal{L}_n(012) = \mathcal{U}_n(012) = \mathcal{I}_n(012), \quad \mathcal{L}_n(102) = \mathcal{U}_n(102).$$

Under the zero-based convention, neither the class of restricted growth functions nor the reuse-growth class contains the other in every length.

Proof. Every adjacent equality is a non-first occurrence, so $\text{lev}(w) \leq \text{rep}(w)$ for every word w . This proves $\mathcal{L}_n \subseteq \mathcal{U}_n$. A nonempty prefix of length $i - 1$ has at most $i - 2$ non-first occurrences. Its reuse bound is therefore at most $i - 1$, proving $\mathcal{U}_n \subseteq \mathcal{I}_n$.

For any word starting with 0, avoiding 012 is equivalent to having a weakly decreasing positive subsequence. An increasing pair of positive entries forms 012 together with the initial zero; conversely, a 012 occurrence contains such a pair. Now let $e \in \mathcal{I}_n(012)$. The all-zero word is in $\mathcal{L}_n$. Otherwise suppose the first positive value a follows $k \geq 1$ zeros. The inversion bound gives $a \leq k$. Both the level bound and the reuse bound after these zeros are k . Every later positive value is at most a , and neither bound decreases, so $e \in \mathcal{L}_n(012)$. Together with the inclusions, this proves the 012 equalities.

It remains to prove the reverse inclusion for 102. Let $w \in \mathcal{U}_n(102)$ and suppose, for a contradiction, that w is not a level sequence. Choose the first position i at which the level bound fails, put $v = w_1 \cdots w_{i-1}$, and set $x = w_i$. Since all earlier positions satisfy the level bound and lev is nondecreasing along prefixes, every entry of v is at most $1 + \text{lev}(v)$. Thus

$$x > 1 + \text{lev}(v) \geq \max(v),$$

so x is a new global maximum. On the other hand, reuse legality gives

$$\text{rep}(v) \geq x - 1 > \text{lev}(v).$$

Hence some repeated position of v is not an adjacent equality. Such a nonadjacent return forces a descent somewhere in v : if a value b reappears after a different preceding value, then either that preceding value is larger than b , or the segment from an earlier occurrence of b down to the preceding value contains an adjacent descent. Let $a > b$ be a decreasing pair in v . Since x is larger than every entry of v , the subsequence a, b, x is order-isomorphic to 102, a contradiction. Thus w satisfies the level bound. Since $\mathcal{L}_n \subseteq \mathcal{U}_n$, we obtain $\mathcal{L}_n(102) = \mathcal{U}_n(102)$.

A zero-based restricted growth function satisfies $w_1 = 0$ and $w_i \leq 1 + \max(w_1, \dots, w_{i-1})$. The word 012 satisfies this rule but not the reuse rule; 002 satisfies the reuse rule but not the restricted growth function rule. $\square$

3 Six single-pattern enumerations

The thirteen packed patterns of length three are:

$$000, 001, 010, 011, 012, 021, 100, 101, 102, 110, 120, 201, 210.$$

For a pattern p , let $\mathcal{U}_n(p)$ be the avoiding subclass and write

$$R_p(x) = \sum_{n \geq 1} |\mathcal{U}_n(p)| x^n.$$

We begin with four short enumerations. The calculations for 021 and 102 then show how restrictions on legal extensions lead to different algebraic series.

3.1 The patterns 000 and 001

Theorem 3.1. *For $n \geq 1$,*

$$|\mathcal{U}_n(000)| = 2^{n-1}, \quad R_{000}(x) = \frac{x}{1 - 2x}.$$

Proof. Every value in a 000-avoiding prefix occurs at most twice. Hence $\text{rep}(w)$ equals the number of values occurring twice. The legal reuse interval has $\text{rep}(w) + 2$ values; exactly $\text{rep}(w)$ cannot be chosen because that would create a third occurrence. Thus every nonempty avoiding prefix has exactly two avoiding extensions. $\square$

Theorem 3.2. For $n \geq 1$,

$$\mathcal{U}_n(001) = \{01^a0^{n-a-1} : 0 \leq a \leq n-1\}.$$

Consequently,

$$|\mathcal{U}_n(001)| = n, \quad R_{001}(x) = \frac{x}{(1-x)^2}.$$

Proof. Before the first repeated entry, $\text{rep} = 0$, so only 0 and 1 can occur. Once some value b occurs twice, a later value greater than b would form 001. In particular, no value at least 2 can occur in an avoider. Since the first entry is 0, the second zero forbids every later 1. This gives the displayed form. Conversely, every displayed word is legal and avoids 001. $\square$

Each displayed word is also a level sequence, so $\mathcal{L}_n(001) = \mathcal{U}_n(001)$. This agrees with the level-sequence enumeration in [4].

3.2 The pattern 011 and involutions

Let I_n denote the number of involutions of $[n]$.

Theorem 3.3. For every $n \geq 1$, $|\mathcal{U}_n(011)| = I_n$. With $|\mathcal{U}_0(011)| = 1$,

$$\sum_{n \geq 0} |\mathcal{U}_n(011)| \frac{x^n}{n!} = \exp(x + x^2/2).$$

Proof. Because the word starts with 0, avoiding 011 means that every positive value occurs at most once. If a prefix contains z zeros and p positive entries, then its length is $z + p$, its number of distinct values is $p + 1$, and its reuse bound is z . Thus a positive value j may occur only after the j th zero.

Number the zero occurrences from left to right. Pair the position of a positive entry j with the position of the j th zero. Every positive value is used at most once, so these pairs are disjoint. Unpaired zero positions are singletons. This gives a partition of $[n]$ into blocks of sizes one and two, or equivalently an involution.

Conversely, in an involution, put zero at every fixed point and at the smaller endpoint of every transposition. Number these zero positions from left to right. At the partner of the j th zero, put j . The partner occurs later, so the reuse bound there is at least j . Each positive value is used once, and the word avoids 011. The two constructions are inverse. The exponential generating function for singletons and pairs is $\exp(x + x^2/2)$. $\square$

For comparison, the inversion-sequence class avoiding 011 is counted by the Bell numbers [7]. Here the reuse bound gives the involution encoding above.

3.3 The pattern 012

Lemma 3.4. A reuse-growth sequence avoids 012 if and only if its positive subsequence is weakly decreasing.

Proof. An increasing pair of positive entries forms 012 with the initial zero. Conversely, every 012 occurrence contains an increasing pair of positive entries. $\square$

By Proposition 2.6, this class is exactly $\mathcal{I}_n(012)$, whose Fibonacci enumeration is known [7]. The following continuation calculation gives the same formula directly from the reuse rule and will also be useful for comparison with 021.

For $\ell \geq 1$, let $G_\ell(x)$ count all continuations of a 012-avoiding prefix whose last positive entry is ℓ , including the empty continuation. The old choices are 0 and ℓ , and every positive $j < \ell$ is new. All these values satisfy the reuse bound. Therefore

$$G_\ell = 1 + x \left(2G_\ell + \sum_{j=1}^{\ell-1} G_j \right),$$

and induction gives

$$G_\ell(x) = \frac{(1-x)^{\ell-1}}{(1-2x)^\ell}. \quad (4)$$

Theorem 3.5. For $n \geq 1$, $|\mathcal{U}_n(012)| = F_{2n-1}$, where $F_0 = 0$ and $F_1 = 1$. Equivalently,

$$R_{012}(x) = \frac{x(1-x)}{1-3x+x^2}.$$

Proof. All-zero words contribute $x/(1-x)$. After $k \geq 1$ initial zeros, the first positive value can be any $j \in \{1, \dots, k\}$. Hence

$$R_{012}(x) = \frac{x}{1-x} + \sum_{k \geq 1} x^{k+1} \sum_{j=1}^k G_j(x).$$

By (4),

$$\sum_{j \geq 1} x^j G_j(x) = \frac{x}{1-3x+x^2}.$$

Interchanging the sums, coefficientwise, gives

$$\sum_{k \geq 1} x^{k+1} \sum_{j=1}^k G_j(x) = \frac{x}{1-x} \sum_{j \geq 1} x^j G_j(x) = \frac{x^2}{(1-x)(1-3x+x^2)}.$$

Adding the all-zero term gives the stated rational function, whose coefficients are F_{2n-1} . $\square$

3.4 The pattern 021

Lemma 3.6. A reuse-growth sequence avoids 021 if and only if its positive subsequence is weakly increasing.

Proof. A decreasing pair of positive entries $a > b > 0$ forms 021 with the initial zero. Conversely, every 021 occurrence contains such a pair. $\square$

Suppose a positive entry has occurred. Let L be the last positive entry and put $s = B(w) - L \geq 0$. The entries $0, L$ are old legal choices, and $L+1, \dots, B(w)$ are new. The succession rule is

$$(s) \longrightarrow 2(s+1), (0), (1), \dots, (s-1). \quad (5)$$

Let $F_s(x)$ count continuations from state s , with the empty one included. Then

$$F_s = 1 + x \left(2F_{s+1} + \sum_{j=0}^{s-1} F_j \right). \quad (6)$$

For $F(u) = \sum_{s \geq 0} F_s u^s$, summation yields

$$[-2x + (1+2x)u - (1+x)u^2]F(u) = u - 2x(1-u)F_0. \quad (7)$$

The kernel root with zero constant term is

$$U(x) = \frac{1 + 2x - \sqrt{1 - 4x - 4x^2}}{2(1 + x)}.$$

Substitution gives $F_0 = U/[2x(1 - U)]$.

Theorem 3.7. *The ordinary generating function is*

$$R_{021}(x) = \frac{1 - 2x^2 - \sqrt{1 - 4x - 4x^2}}{2(1 - x + x^2)}.$$

Equivalently,

$$(1 - x + x^2)R_{021}^2 + (2x^2 - 1)R_{021} + x + x^2 = 0.$$

The counts for $n = 1, 2, \dots$ begin 1, 2, 5, 15, 50, 179, 673, 2622, $\dots$

Proof. After $k \geq 1$ initial zeros, the first positive value $j \in \{1, \dots, k\}$ leaves state $k - j$. Thus

$$R_{021}(x) = \frac{x}{1 - x} + \frac{x^2}{1 - x}F(x).$$

Specializing (7) at $u = x$ gives

$$F(x) = \frac{2(1 - x)F_0 - 1}{1 - x + x^2}.$$

Substitution of the expression for F_0 and then for U gives the stated formula. Squaring its defining radical gives the quadratic equation; its solution with zero constant term is the one above. $\square$

The characterization in Lemma 3.6 also holds for inversion sequences, where it gives the large Schröder numbers [7]. The classes are not equal: $012 \in \mathcal{I}_3(021)$ but $012 \notin \mathcal{U}_3$. Thus the same order condition does not remove the difference between the growth bounds.

3.5 The pattern 102

By Proposition 2.6, the reuse-growth and level-sequence avoidance classes are identical for this pattern. Mansour explicitly gives the single-pattern level-sequence generating functions for 001 and 012, but not for 102 [4, Section 3]; the theorem below therefore gives the enumeration of $\mathcal{L}_n(102)$ as well. For comparison, Mansour and Shattuck's 102-avoiding inversion sequences have generating function $A = 1 + (x - x^2)A^3$ [8]. We derive a different cubic for the common reuse/level class.

Let $w = w_1 \cdots w_m \in \mathcal{U}_m(102)$ contain a decreasing pair. Define its *cap* by

$$L(w) = \min\{w_i : \text{there is } j > i \text{ with } w_i > w_j\}.$$

The positions i and j need not be adjacent. Appending c creates 102 exactly when some decreasing pair $w_i > w_j$ has $w_i < c$, that is, when $c > L(w)$. Thus the avoiding extensions are exactly $0, 1, \dots, L(w)$. These values satisfy the reuse bound because $L(w)$ has already occurred.

For example, $w = 000231$ has cap 2, although its only adjacent descent is $3 > 1$. Appending 3 creates the subsequence $2, 1, 3 \sim 102$. Below we write $L = L(w)$.

Record used and unused values in this interval, in increasing numerical order, by O and N. The record has the form

$$\text{ON}^{a_1}\text{ON}^{a_2}\cdots\text{ON}^{a_k}\text{O}, \quad a_1, \dots, a_k \geq 0, \quad k \geq 1.$$

Its endpoints are used: 0 is used initially and L is the larger entry of a decreasing pair. Let $C_{a_1, \dots, a_k}(x)$ count all legal continuations of this capped state, including the empty continuation. If its old values are $c_0 < \dots < c_k = L$, choosing c_k or c_{k-1} leaves the record unchanged. Choosing an earlier old value truncates the record at the old value immediately above it. Choosing an unused value splits its gap, and the cap becomes the old value immediately above that gap. These cases give

$$(1-2x)C_{a_1, \dots, a_k} = 1 + x \sum_{j=1}^{k-1} C_{a_1, \dots, a_j} + x \sum_{j=1}^k \sum_{h=1}^{a_j} C_{a_1, \dots, a_{j-1}, h-1, a_j-h}. \quad (8)$$

Define $D_a(x)$ by

$$(1-2x)D_a = 1 + x(1-x) \sum_{h=1}^a D_{h-1}D_{a-h}, \quad D_0 = \frac{1}{1-2x}. \quad (9)$$

Lemma 3.8. For $k \geq 1$,

$$C_{a_1, \dots, a_k} = (1-x)^{k-1} \prod_{j=1}^k D_{a_j}. \quad (10)$$

Proof. Put $P_j = (1-x)^{j-1} \prod_{i=1}^j D_{a_i}$ for $j \geq 1$, and set $P_0 = 1/(1-x)$. Substitute the proposed product into the right-hand side of (8). The j th inner sum, including its factor x , becomes

$$(1-x)^j \prod_{i < j} D_{a_i} x \sum_{h=1}^{a_j} D_{h-1}D_{a_j-h} = (1-2x)P_j - (1-x)P_{j-1},$$

by (9). Summing over j gives

$$(1-2x)P_k - x \sum_{j=1}^{k-1} P_j = 1.$$

The other terms on the right of (8) cancel the last two terms, leaving $(1-2x)P_k$. The recurrence determines every coefficient from coefficients of lower degree, even though some state indices may coincide. Hence the product is the required continuation series. $\square$

For a Catalan form of D_a , put

$$\zeta = \frac{x(1-x)}{(1-2x)^2}, \quad Q_a(\zeta) = (1-2x)D_a(x).$$

Equation (9) becomes

$$Q_0 = 1, \quad Q_a = 1 + \zeta \sum_{j=0}^{a-1} Q_j Q_{a-1-j}.$$

Thus $Q(u) = \sum_{a \geq 0} Q_a u^a$ satisfies

$$Q(u) = \frac{1}{1-u} + \zeta u Q(u)^2.$$

Writing $\mathcal{C}(t) = \sum_{j \geq 0} C_j t^j = (1 - \sqrt{1-4t})/(2t)$ for the Catalan series gives

$$Q(u) = \frac{1}{1-u} \mathcal{C}\left(\frac{\zeta u}{1-u}\right), \quad Q_a(\zeta) = \sum_{j=0}^a \binom{a}{j} C_j \zeta^j. \quad (11)$$

It remains to include prefixes with no decreasing pair. Such a prefix is weakly increasing. Let its internal gaps be $a_1, \dots, a_k$ and let r be the number of unused legal values above its maximum. Write $U_{a_1, \dots, a_k; r}(x)$ for its continuation series. Set $C_\emptyset = 1/(1-x)$ when there is no internal gap. By (8), the contribution of all downward choices, including their factors x , is $(1-x)C_{a_1, \dots, a_k} - 1$. Repeating the maximum increases r by one. Choosing the h th unused value above it adds an internal gap of size $h-1$ and leaves $r-h$ unused values above the new maximum. Therefore

$$U_{a_1, \dots, a_k; r} = (1-x)C_{a_1, \dots, a_k} + xU_{a_1, \dots, a_k; r+1} + x \sum_{h=1}^r U_{a_1, \dots, a_k, h-1; r-h}. \quad (12)$$

Adjoining a gap of size j multiplies the capped factor by $H_j = (1-x)D_j$. Substitution in (12), followed by coefficientwise uniqueness, yields

$$U_{a_1, \dots, a_k; r} = C_{a_1, \dots, a_k} T_r,$$

where

$$T_r = 1 - x + xT_{r+1} + x \sum_{j=0}^{r-1} H_j T_{r-1-j}. \quad (13)$$

Let $T(u) = \sum_{r \geq 0} T_r u^r$ and $H(u) = \sum_{j \geq 0} H_j u^j$. Then

$$[u - x - xu^2 H(u)]T(u) = \frac{u(1-x)}{1-u} - xT_0, \quad (14)$$

where

$$H(u) = \frac{1-x}{1-2x} \frac{1}{1-u} \mathcal{C}\left(\frac{\zeta u}{1-u}\right).$$

The kernel has a unique root $U(x) \in x\mathbb{Q}[[x]]$, since at $x=0$ it is u . Substituting this root gives

$$T_0 = \frac{U(1-x)}{x(1-U)}. \quad (15)$$

The initial word 0 has no internal gap and one unused value above its maximum. Hence, for $A_{102}(x) = 1 + \sum_{n \geq 1} |\mathcal{U}_n(102)|x^n$,

$$A_{102} = 1 + xU_{\emptyset;1} = 1 + \frac{xT_1}{1-x} = \frac{T_0}{1-x} = \frac{U}{x(1-U)},$$

where the second-to-last equality follows from (13) at $r=0$.

Theorem 3.9. *The series A_{102} is the unique formal power series with constant term 1 satisfying*

$$A_{102} = 1 + x(1-x)A_{102}^2 + x^2(1-x)A_{102}^3. \quad (16)$$

Its coefficients from x^0 onward are 1, 1, 2, 5, 15, 49, 169, 604, 2216, 8296, \dots

Proof. The kernel identity $U = x + xU^2H(U)$ gives

$$\mathcal{C}\left(\frac{\zeta U}{1-U}\right) = \frac{(1-2x)(1-U)(U-x)}{xU^2(1-x)}.$$

Substitute this expression into the Catalan identity $C = 1 + yC^2$, with $y = \zeta U/(1-U)$. Clearing formal factors gives

$$(1+x)U^3 - (3+2x)U^2 + (1+3x)U - x = 0.$$

Since $U = xA_{102}/(1 + xA_{102})$, substitution yields (16). Its coefficient of x^n is determined by earlier coefficients, so the solution with constant term 1 is unique. $\square$

Corollary 3.10 (Lagrange form). *For $n \geq 0$, with $|\mathcal{U}_0(102)| = 1$,*

$$|\mathcal{U}_n(102)| = \frac{1}{n+1} [t^n] \left(\frac{1-t^2-t^3}{1-t-t^2} \right)^{n+1}.$$

Proof. Put $Y = xA_{102}$. Equation (16) becomes

$$Y = x\varphi(Y), \quad \varphi(t) = \frac{1-t^2-t^3}{1-t-t^2}.$$

Lagrange inversion gives $[x^{n+1}]Y = (n+1)^{-1} [t^n]\varphi(t)^{n+1}$, which is the required coefficient of A_{102} . $\square$

Corollary 3.11. *Let τ be the unique root in $(0, 1)$ of*

$$2\tau^3 - 2\tau^2 - 2\tau + 1 = 0,$$

and put

$$\rho = \frac{\tau}{\varphi(\tau)} = \frac{\tau(1-\tau-\tau^2)}{1-\tau^2-\tau^3}.$$

Then $\tau \approx 0.4030317168$, $\rho \approx 0.2268243265$, and

$$|\mathcal{U}_n(102)| \sim \kappa \rho^{-n} n^{-3/2}, \quad \kappa = \frac{1}{2\rho\sqrt{\pi}} \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}} \approx 0.3982359746.$$

Proof. The series φ is analytic before its first positive pole $(\sqrt{5}-1)/2$, and its coefficients are nonnegative. Indeed,

$$\varphi(t) = 1 + t + \frac{t^2}{1-t-t^2}.$$

The solution $Y = x\varphi(Y)$ is aperiodic; for example, $[x]Y = [x^2]Y = 1$. The characteristic equation $\tau\varphi'(\tau) = \varphi(\tau)$ reduces to the displayed cubic. That cubic is strictly decreasing on $(0, 1)$, positive at 0, and negative at $1/2$. Its unique root lies before the pole of φ . The smooth implicit-function schema [14, Chapter VII] gives

$$Y(x) = \tau - \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}} \sqrt{1-x/\rho} + O(1-x/\rho).$$

Since $A_{102} = Y/x$, square-root transfer [14, Chapter VI] gives the assertion. $\square$

4 Exact extension rules for four repeated-letter patterns

For the patterns 010, 100, 101, 110, we use the following record. Given an avoiding prefix w , list, in increasing order, all values in $\{0, \dots, B(w)\}$ that may be appended without creating the pattern. Mark a value O if it has already appeared and N if it has not. The resulting *binary state word* records only currently legal values; a value forbidden by an occurrence already present in the prefix never becomes legal again. The initial word 0 has state ON. Multiple choices leading to the same state retain their multiplicities.

For a binary word S and a chosen position i , define $L_i(S)$ by replacing the chosen symbol with O and deleting every O strictly to its left. Define $R_i(S)$ by deleting every O strictly to its right instead. All other N symbols are retained. Put $\varepsilon_i = 1$ if the chosen symbol was O and 0 if it was N.

Proposition 4.1. *The exact successor rules are*

$$010 : S \xrightarrow{i} L_i(S)N^{\varepsilon_i}, \quad 101 : S \xrightarrow{i} R_i(S)N^{\varepsilon_i}.$$

Every position in the state word gives one legal extension.

Proof. For 010, an old value a is reusable precisely when no earlier occurrence of a has a larger value after it. Once such a larger value occurs, a is permanently forbidden. An unused value is legal at its first occurrence. Appending a makes exactly the currently reusable old values below a unavailable, while a itself remains reusable. This is $L_i(S)$. An old choice increases the bound by one and adds a terminal unused value.

For 101, an old value a is reusable precisely when no smaller entry has appeared after an earlier occurrence of a . Appending a therefore makes exactly the currently reusable old values above a unavailable. Again, an old choice adds a terminal unused value. This is the stated R_i rule. $\square$

For 100, an old value is reusable exactly when none of its occurrences has a larger predecessor. The current maximum is always reusable and is the rightmost O.

Proposition 4.2. *Let $m = m(S)$ be the position of the rightmost O in a 100 state. Its successors are as follows.*

- (i) *If $i < m$, delete the chosen symbol and append one terminal N exactly when it was O.*
- (ii) *If $i = m$, the successor is SN.*
- (iii) *If $i > m$, replace the chosen N by O and make no other change.*

Proof. A new occurrence ending at a has the form baa with $b > a$. If a legal choice a lies below the current maximum M , its newly appended occurrence has M as a larger predecessor. Thus it cannot be used again, whether it was old or new, and its symbol is deleted. An old choice still increases the bound and adds a terminal N. Repeating M leaves it reusable and adds one terminal N. A new value above M becomes the new maximum and is reusable, without increasing the bound. No other old value changes status at this step. $\square$

For 110, put

$$Q(w) = \max(\{a : a \text{ occurs at least twice in } w\} \cup \{0\}).$$

Proposition 4.3. *A reuse-legal extension $a \leq B(w)$ preserves 110-avoidance exactly when $a \geq Q(w)$. The binary state records the interval $[Q(w), B(w)]$. Choosing a N changes it to O. Choosing an O at position i deletes all symbols to its left and appends a terminal N:*

$$S \xrightarrow{i} O S_{i+1} \cdots S_{|S|} N.$$

Proof. Appending a creates 110 exactly when some value greater than a has already occurred twice, or equivalently when $a < Q(w)$. A new choice does not change the threshold. An old legal choice becomes repeated; since $a \geq Q(w)$, the new threshold is a . All smaller values are deleted, and the increase of the reuse bound contributes a terminal N. $\square$

4.1 Weak-composition states for 101

Zero cannot become illegal under 101-avoidance. Hence every state begins with O and has a unique representation

$$ON^{a_1}ON^{a_2}\cdots ON^{a_k}, \quad a_1, \dots, a_k \geq 0.$$

Identify this state with the weak composition $a = (a_1, \dots, a_k)$. The initial state is (1).

Proposition 4.4. *Choosing the j th old value gives*

$$(a_1, \dots, a_k) \longrightarrow \left(a_1, \dots, a_{j-1}, 1 + \sum_{i=j}^k a_i \right), \quad 1 \leq j \leq k.$$

Choosing the h th unused value in the j th gap gives

$$(a_1, \dots, a_k) \longrightarrow \left(a_1, \dots, a_{j-1}, h-1, a_j-h + \sum_{i=j+1}^k a_i \right), \quad 1 \leq h \leq a_j.$$

The indexed transitions retain their multiplicities when their resulting states coincide.

Proof. The 101 rule deletes old values to the right of the chosen value but retains unused values. For an old choice, the unused gaps from its block onward merge, and an additional terminal unused value is added. For a new choice, the $h-1$ unused values before it remain in the preceding gap, while those after it merge with all later unused gaps. These are exactly the two formulas. $\square$

Define

$$\omega(a) = a_1 + \dots + a_k + 2(k-1).$$

Theorem 4.5. *A weak composition a occurs as a 101 state after a word of length n if and only if*

$$a \neq (0), \quad \omega(a) \leq n, \quad \omega(a) \equiv n \pmod{2}.$$

The number s_n of distinct states at level n is $F_{n+2} - 1$.

Proof. The initial state (1) has weight 1. In either transition, if the chosen value lies in block j of a state with k parts, then

$$\omega(\text{child}) = \omega(a) + 1 - 2(k-j).$$

The three asserted conditions are therefore preserved, with n increased by one; (0) is never produced.

Conversely, let $b = (b_1, \dots, b_\ell)$ satisfy the conditions at level $n+1$. If $b_\ell > 0$ and $b \neq (1)$, decrease its last part by one and choose the last old value. If $b_\ell = 0$, then $\ell \geq 2$; delete that zero, increase the new last part by one, and choose the last unused value in that gap. In either case the parent has weight $\omega(b) - 1$, is not (0), and gives the desired state by Proposition 4.4. The exceptional state (1) is the root at level 1; at every later odd level it is obtained from (0, 0) by choosing the first old value. Induction proves the characterization.

The generating function for all weak compositions by their minimum possible weight, excluding (0), is

$$\sum_{k \geq 1} \frac{z^{2(k-1)}}{(1-z)^k} - 1 = \frac{1}{1-z-z^2} - 1.$$

A state of weight c occurs at levels $c, c+2, c+4, \dots$. Hence

$$\sum_{n \geq 1} s_n z^n = \frac{1}{1-z^2} \left(\frac{1}{1-z-z^2} - 1 \right) = \frac{z}{(1-z)(1-z-z^2)},$$

whose coefficient is $F_{n+2} - 1$. $\square$

4.2 A common factorial lower bound

Theorem 4.6. *For each $p \in \{010, 100, 101, 110\}$ and $n \geq 1$,*

$$|\mathcal{U}_n(p)| \geq \lfloor n/2 \rfloor!.$$

In particular, each class is superexponential and its ordinary generating series has radius of convergence zero.

Proof. Put $m = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$. Begin with 0^k and append an arbitrary permutation π of $1, \dots, m$. The reuse bound after the zeros is k , and every appended value is new, so that bound stays fixed. As $m \leq k$, every word $0^k\pi$ is legal.

Its only repeated value is zero, and all zeros precede all positive entries. Therefore 010 cannot occur, since it would need a positive entry between two zeros; 100 cannot occur, since it would need a positive entry before two zeros; and neither 101 nor 110 can use zero as its larger repeated value. No positive value repeats. The $m!$ permutations give distinct words avoiding all four patterns. Finally, $(\lfloor n/2 \rfloor!)^{1/n} \rightarrow \infty$, so the Cauchy–Hadamard formula gives radius zero. $\square$

The same words are inversion sequences by Proposition 2.6, so this lower bound also applies to the corresponding inversion-sequence avoidance classes.

5 The patterns 120, 201, 210 and two intersections

We now give exact extension tests for 120, 210, 201. As in Section 4, an O/N record lists currently legal values in numerical order. The reuse bound $B(w) = 1 + \text{rep}(w)$ is distinct from a threshold imposed by avoidance.

5.1 The single patterns 120, 210, and 201

For a 120-avoiding prefix, put

$$A(w) = \max(\{w_i : \exists j > i, w_i < w_j\} \cup \{0\}).$$

Proposition 5.1. *A reuse-legal extension $a \leq B(w)$ preserves 120-avoidance exactly when $a \geq A(w)$. After appending a , the new threshold is the maximum of $A(w)$ and the largest previously used value smaller than a , when such a value exists.*

Proof. A new occurrence must end at a and has the form $w_i w_j a$ with $i < j$ and $a < w_i < w_j$. Such a pair exists exactly when $a < A(w)$. The only newly created increasing pairs end at a , and their smaller values are precisely the previously used values below a . This gives the update. $\square$

In the binary record, the update deletes all symbols strictly to the left of the rightmost old value below the chosen value, when there is one. A chosen N becomes O, and an old choice appends a terminal N.

For a 210-avoiding prefix, put

$$D(w) = \max(\{w_j : \exists i < j, w_i > w_j\} \cup \{0\}).$$

Proposition 5.2. *A reuse-legal extension a preserves 210-avoidance exactly when $a \geq D(w)$. If $a < M = \max(w)$, the new threshold is a ; otherwise it is unchanged.*

Proof. A new occurrence ending at a requires a previous decreasing pair $w_i > w_j > a$, which exists exactly when $a < D(w)$. If $a < M$, the pair M, a is decreasing, and legality gives $a \geq D(w)$. If $a \geq M$, no new decreasing pair is created. $\square$

Thus, when the chosen value is below M , delete all smaller symbols in the binary record. Again change a new value to old and append a new terminal unused value for an old choice.

For 201, define the set of legal separating values

$$\Sigma(w) = \{a \in \{0, \dots, B(w)\} : \nexists i < j \text{ with } w_i > a > w_j\}.$$

Proposition 5.3. *A reuse-legal extension a preserves 201-avoidance exactly when $a \in \Sigma(w)$. If $M = \max(w)$ and $a < M$, then the old legal set loses exactly the values in (a, M) . If $a \geq M$, no existing legal value is lost. In either case, an old choice also adds the new top value $B(w) + 1$.*

Proof. A new 201 occurrence ending at a has the form $w_i w_j a$ with $w_i > a > w_j$. This is exactly what the definition of $\Sigma(w)$ excludes. If $a < M$, the new pair M, a forbids every future value strictly between them. Any new descending pair ending at a forbids only values in that same interval, so there are no other deletions. If $a \geq M$, no descending pair is created. An old choice raises the reuse bound by one; the new top value exceeds every used value and is legal. $\square$

Remark 5.4. For 120 and 210, the legal set inside the reuse interval is a tail interval. For 201, it can have holes. These tests give exact generating rules on ordered O/N records.

5.2 The pair {201,210}

Inversion sequences avoiding 201 and 210 were enumerated by Chen and Lin [16, Theorem 4.8]. Alternative derivations were given by Pantone [12] and by Testart [13], who constructs inversion sequences from right to left. The rule below instead appends entries under the reuse-growth bound.

If a decreasing pair $M > a$ has occurred, every later value b must satisfy $b = a$ or $b \geq M$: the triple is 210 when $b < a$, and 201 when $a < b < M$. This restriction depends only on the forbidden patterns and is also used for inversion sequences in [16, 12]. For reuse-growth words it gives the following three-parameter rule.

Let M be the maximum of an avoiding prefix. Write (a, b, r) for the numbers of old legal values below M , unused legal values below M , and unused legal values above M and at most $B(w)$, respectively. Keep M itself separate. The initial state is $(0, 0, 1)$.

Proposition 5.5. *The succession rule, with multiplicities written as superscripts, is*

$$(a, b, r) \longrightarrow (1, 0, r+1)^a, (1, 0, r)^b, (a, b, r+1), (a+1, b+h-1, r-h)_{1 \leq h \leq r}. \quad (17)$$

Proof. Choose a legal value $c < M$. The new decreasing pair $M > c$ makes c the only legal value below M . If c is old, the reuse bound rises by one, giving $(1, 0, r+1)$, with a choices. If c is new, the bound is unchanged, giving $(1, 0, r)$, with b choices.

Repeating M leaves the legal values below it unchanged and creates one new unused top value, giving $(a, b, r+1)$. Finally, choosing the h th unused value above M makes it the new maximum. No occurrence of either pattern can end at a new maximum. The old maximum becomes an old legal value below it; the $h-1$ skipped values become unused legal values below it; and $r-h$ unused values remain above it. This gives the final family of children. $\square$

Let

$$H(x; u, v, z) = \sum x^m u^a v^b z^r,$$

where the sum is over words reached after m extensions of 0, with their states, so the root contributes z . We suppress x when displaying the catalytic variables. The four kinds of children

give

$$H(u, v, z) = z + x \left[uzH_u(1, 1, z) + uH_v(1, 1, z) + zH(u, v, z) + \frac{u}{z-v}(H(u, v, z) - H(u, v, v)) \right]. \quad (18)$$

Define

$$\alpha(x) = \frac{1 + 2x - \sqrt{(1+2x)(1-6x)}}{4x}, \quad \beta(x) = \frac{1 + x - \sqrt{(1+x)(1-3x)}}{2x}. \quad (19)$$

Both have constant term 1.

Theorem 5.6. *Let $T(x) = \sum_{n \geq 1} |\mathcal{U}_n(201, 210)|x^{n-1}$. Then*

$$T(x) = \frac{(1+2x)\beta(x)(\alpha(x)-1) + 2(1+x)}{x(1-2x)\beta(x)(\alpha(x)-1) + 2(1-x^2)}. \quad (20)$$

The counts for $n = 1, 2, \dots$ begin

$$1, 2, 5, 16, 59, 243, 1076, 5023, 24346, 121372, \dots$$

Proof. Set

$$F(z) = H(1, 1, z), \quad P(z) = H_u(1, 1, z), \quad Q(z) = H_v(1, 1, z),$$

and write $T = F(1)$, $R = F'(1)$, $P_1 = P(1)$, $Q_1 = Q(1)$. Specializing (18) gives

$$F(z) = z + x \left[zP(z) + Q(z) + zF(z) + \frac{F(z) - T}{z-1} \right]. \quad (21)$$

For the first a -moment, an old-below choice has child weight 1, a repeat of the maximum has child weight a , and an upper-new choice has child weight $a+1$. Summing the contributions gives

$$P(z) = x \left[2zP(z) + Q(z) + \frac{P(z) + F(z) - P_1 - T}{z-1} \right]. \quad (22)$$

For the first b -moment, only repetition of the maximum and upper-new choices contribute. Since

$$\sum_{h=1}^r (h-1)z^{r-h} = \frac{z^r - 1 - r(z-1)}{(z-1)^2},$$

we get

$$Q(z) = x \left[zQ(z) + \frac{Q(z) - Q_1}{z-1} + \frac{F(z) - T - (z-1)R}{(z-1)^2} \right]. \quad (23)$$

All quotients at $z = 1$ are interpreted by their formal limits. In particular,

$$P_1 + Q_1 + R = \frac{(1-x)T - 1}{x}. \quad (24)$$

Eliminate $Q(z)$ between (21) and (22). Multiplication by $z-1$ and collection of terms give

$$K_\beta(z)P(z) = xP_1 + (K_\beta(z) - x)F(z) + z(z-1), \quad (25)$$

where $K_\beta(z) = xz^2 - (1+x)z + (1+x)$. Its formal power-series root is $\beta(x)$; its other root has a pole at $x = 0$. At each fixed power of x , the coefficients of $F(z)$ and $P(z)$ are polynomials in z , so substitution is legitimate. It yields

$$F(\beta) = P_1 + \frac{\beta(\beta-1)}{x}. \quad (26)$$

For the second elimination, write

$$Q(z) = \frac{F(z) - z}{x} - zP(z) - zF(z) - \frac{F(z) - T}{z - 1}, \quad (27)$$

$$P(z) = \frac{xP_1 + (K_\beta(z) - x)F(z) + z(z - 1)}{K_\beta(z)}. \quad (28)$$

Substitution in (23), followed by clearing denominators and cancelling the common factor, gives

$$\begin{aligned} 0 &= (2x^2z^3 - 2x^2z^2 + 2x^2z - 3xz^2 + 3xz - 2x + z - 1)F(z) \\ &\quad + x^2zP_1 + x^2Q_1 + x^2R - x^2zT + xT \\ &\quad + 2xz^3 - 2xz^2 + xz - z^2 + z. \end{aligned}$$

The coefficient of $F(z)$ factors as $(xz - 1)K_\alpha(z)$, where

$$K_\alpha(z) = 2xz^2 - (1 + 2x)z + (1 + 2x).$$

Using (24) to replace $Q_1 + R$ leaves

$$\begin{aligned} 0 &= (xz - 1)K_\alpha(z)F(z) + x^2(z - 1)P_1 + x(2 - x - xz)T \\ &\quad + 2xz^3 - 2xz^2 + xz - x - z^2 + z. \end{aligned} \quad (29)$$

Set $z = \alpha(x)$ and use $2x\alpha^2 = (1 + 2x)\alpha - (1 + 2x)$. The result is

$$x(\alpha - 1)P_1 + (2 - x(\alpha + 1))T = \alpha + 1. \quad (30)$$

Next set $z = \beta$ in (29), use (26), and reduce with $x\beta^2 = (1 + x)\beta - (1 + x)$. This gives

$$(x(\beta - 1) - 1)P_1 + (2 - x(\beta + 1))T = \beta + 1. \quad (31)$$

The determinant of these two linear equations is nonzero at $x = 0$. Solving for T yields

$$T = -\frac{2\alpha x + \alpha - 2\beta x + 1}{2\alpha x^2 - \alpha x - 2\beta x^2 + 2\beta x + x - 2}.$$

The two quadratic identities for α and β transform this expression into (20). Since α and β are algebraic and T is a rational expression in them, T is algebraic. The exponent of x counts extensions of the root word, which explains the shift x^{n-1} . $\square$

Corollary 5.7. *As $n \rightarrow \infty$,*

$$|\mathcal{U}_n(201, 210)| \sim \frac{3(511\sqrt{3} - 171\sqrt{7})}{13778\sqrt{\pi}} 6^n n^{-3/2}.$$

Proof. The function β is analytic in $|x| < 1/3$, and the first positive branch point of α is $\rho = 1/6$. The denominator in (20) does not vanish in $|x| \leq \rho$. To see this, put $A = \alpha - 1$ and $B = \beta - 1$. Their equations are

$$A = 2x(1 + A + A^2), \quad B = x(1 + B + B^2),$$

so they have nonnegative coefficients. Thus for $|x| \leq 1/6$,

$$|\alpha(x) - 1| \leq 1, \quad |\beta(x)| \leq \beta(1/6) = \frac{7 - \sqrt{21}}{2}.$$

If $D(x)$ is the denominator of (20), the triangle inequality gives

$$|D(x)| \geq 2(1 - |x|^2) - |x|(1 + 2|x|)|\beta(x)||\alpha(x) - 1| \geq \frac{21 + 2\sqrt{21}}{18} > 0.$$

Hence the only possible singularity on the circle $|x| = \rho$ is the branch point of α at ρ .

Put $X = 1 - 6x$. At $x = \rho$, $\alpha(\rho) = 2$ and $\beta(\rho) = (7 - \sqrt{21})/2$, and

$$\alpha(x) = 2 - \sqrt{3} X^{1/2} + O(X), \quad \beta(x) = \beta(\rho) + O(X).$$

Substitution into (20) gives

$$T(x) = T(\rho) - \frac{18(511\sqrt{3} - 171\sqrt{7})}{6889} X^{1/2} + O(X).$$

Its square-root coefficient is nonzero, so the singularity is not cancelled. Square-root transfer [14, Chapter VI], applied to $[x^{n-1}]T$, gives the stated constant and growth rate. $\square$

5.3 The pair {120,210}

Proposition 5.8. *Suppose a prefix avoiding 120 and 210 uses at least two distinct values, and let $S < M$ be its two largest used values. A reuse-legal extension c preserves both avoidances exactly when $c \geq S$. With*

$$a = M - S - 1, \quad r = B(w) - M,$$

the succession rule is

$$(a, r) \longrightarrow 2(a, r + 1) + \sum_{j=0}^{a-1} (j, r) + \sum_{h=1}^r (h - 1, r - h). \quad (32)$$

Proof. If $c < S$, the first occurrences of S and M , followed by c , form 120 or 210 according to their order. Conversely, if $c \geq S$, at most one used value is greater than c . Both forbidden patterns require two earlier distinct values greater than their final entry, so neither can be completed.

All a values between S and M , and all r legal values above M , are unused. Repeating S or M raises the bound by one without changing the two maxima, giving two copies of $(a, r + 1)$. A new value between them becomes the second maximum; the remaining gap above it takes each size $0, \dots, a - 1$ once. Choosing the h th unused value above M makes the old maximum the second maximum, giving $(h - 1, r - h)$. $\square$

Prefixes with only one distinct value are the words 0^k , with $B(0^k) = k$. Appending another zero continues this initial chain. Appending $j \in \{1, \dots, k\}$ enters the two-parameter rule in state $(j - 1, k - j)$. These initial conditions together with (32) give an exact generating tree, but no closed ordinary generating function is derived here.

Remark 5.9. For the single patterns 120, 201, 210, the descriptions above retain ordered binary records of unbounded length. For the two pairs, a fixed number of integer parameters suffices.

6 Open problems

Closed generating functions are still missing for seven of the thirteen single patterns. For 010, 100, 101, and 110, the factorial lower bound determines neither the correct asymptotic scale nor a useful exponential generating function. For 120, 201, and 210, the ordered O/N records in Section 5 give exact recursions, but we do not know a closed form for the counting sequence or its asymptotic growth.

The pair {120, 210} is a more specific problem. Proposition 5.8 gives a two-parameter succession rule with explicit initial conditions. It remains to solve the corresponding functional equation, or to show that a different description is more effective.

The insertion bijections also leave a bijective question. It would be useful to describe the images under Φ_n or Θ_n of a nontrivial avoidance class such as $\mathcal{U}_n(102)$ or $\mathcal{U}_n(201, 210)$. Such a description could give an independent explanation of one of the generating functions obtained here.

For a fixed pattern p and a reachable state S in one of the exact descriptions above, let $c_r(S)$ be the number of legal continuations of length r , counted with the transition multiplicities, and put

$$F_S(t) = \sum_{r \geq 0} c_r(S)t^r.$$

Which pairs of distinct reachable states S, S' satisfy $F_S(t) = F_{S'}(t)$? Characterizing this equality would determine when different state records give the same continuation counts.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used OpenAI ChatGPT for proof auditing, algebraic and computational cross-checks, and language editing. Computational checks were used to test enumerations and implementations, not as substitutes for proofs.

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